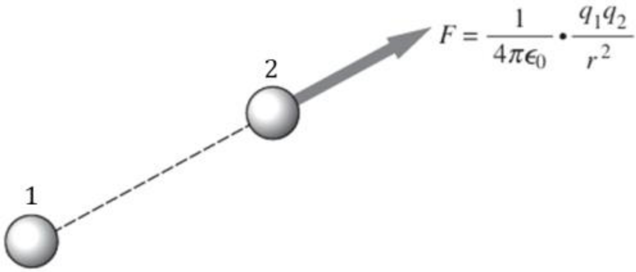
—Chapter 2—

The Electric Field and Potential

2-1 The Electric Field

A. COULOMB'S LAW

(1) Coulomb's force

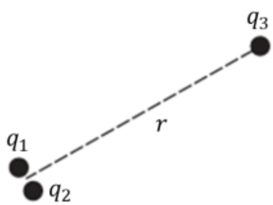


The interaction between electric charges at rest is

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{r}$$

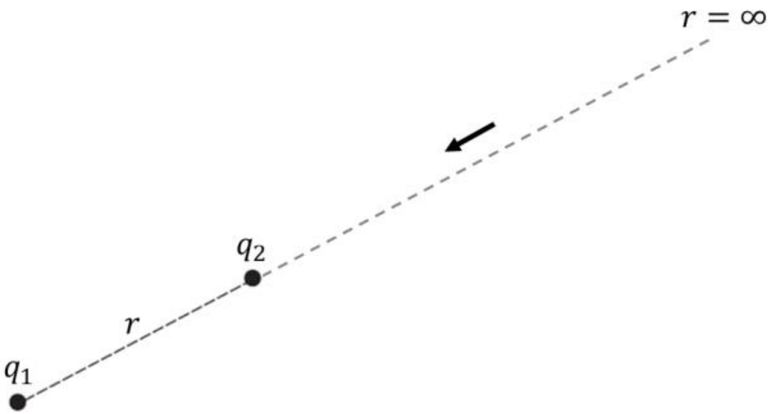
where $\hat{r}$ is the unit vector in the direction from charge 1 to charge 2, and $\vec{F}$ is the force acting on charge 2. The unit vector $\hat{r}$ shows that the force is parallel to the line joining the charges.

(2) Coulomb's force is additive:



$$\vec{F} = \vec{F}_{13} + \vec{F}_{23} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_3}{r^2} \hat{r} + \frac{1}{4\pi\epsilon_0} \frac{q_2 q_3}{r^2} \hat{r} = \frac{(q_1 + q_2) q_3}{4\pi\epsilon_0 r^2} \hat{r}$$

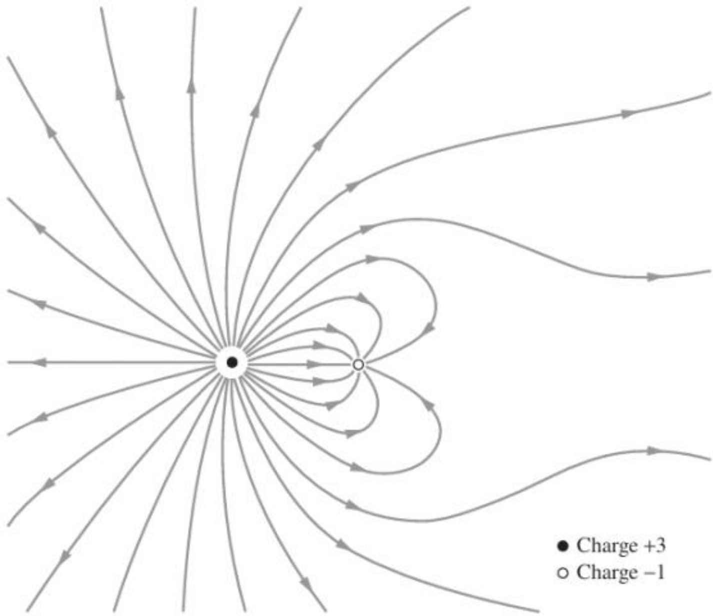
(3) The work done by the force



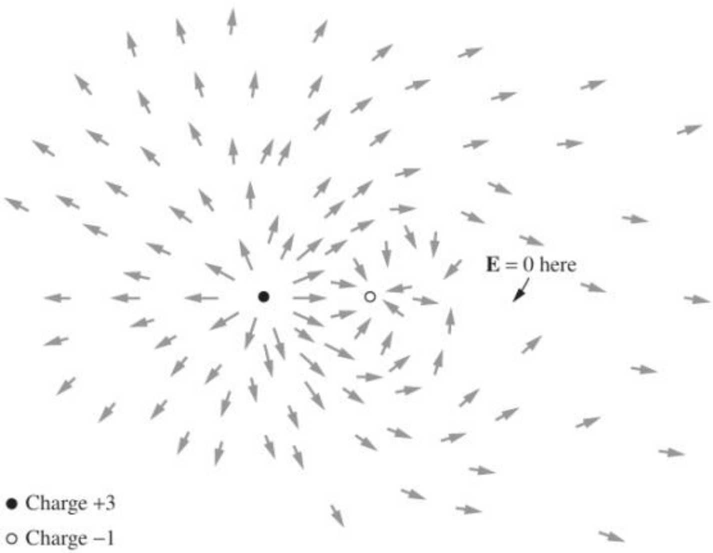
$$
\begin{aligned}
W &= \int_{\infty}^{r} \vec{F}_{\text{ext}} \cdot d\vec{r}' \\
&= \int_{\infty}^{r} -\vec{F} \cdot d\vec{r}' \\
&= -\int_{\infty}^{r} \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r'^2} dr' \\
&= \frac{q_1 q_2}{4\pi\epsilon_0} \left(\frac{1}{r'}\right)_{\infty}^{r} \\
&= \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r}
\end{aligned}
$$

B. THE ELECTRIC FIELD

(1) Faraday proposed: line of force



(2) Maxwell proposed: vector field



Assume that there is a fixed charge in the space. We called it the **source** charge. Thus, the magnitude and direction of the electric field produced by the source charge at a point in space is

$$\vec{E} = \frac{\vec{F}}{q_2}$$

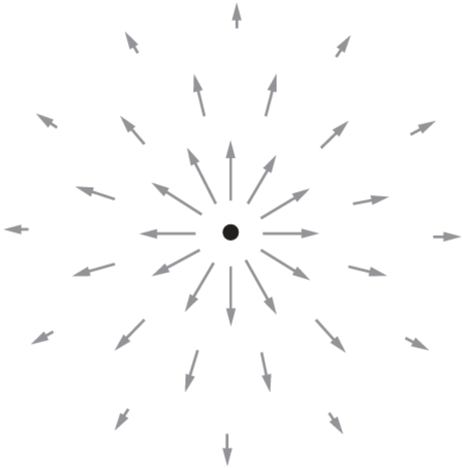
(3) Point charge

The Coulomb's force between two point charges q_1 and q_2:

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$$

The electric field of a point charge (source) at a point in space is

$$\vec{E} = \frac{\vec{F}}{q_2} = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r} \rightarrow \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$



Applying Gauss's divergence theorem to a point charge

$$\Phi = \iiint_{\mathcal{V}} \nabla\cdot\vec{E}\, d\tau = \oiint_{\mathcal{S}} \vec{E}\cdot d\vec{a}$$

L.H.S.:

$$\nabla\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}1 = 0$$

The divergence is zero everywhere except at the origin because as $r \rightarrow 0$, $1/r^2 \rightarrow \infty$ grows faster than $r^2 \rightarrow 0$.

We thus define

$$\nabla\cdot\frac{1}{r^2}\hat{r} = 4\pi\delta^3(r)$$

and obtain

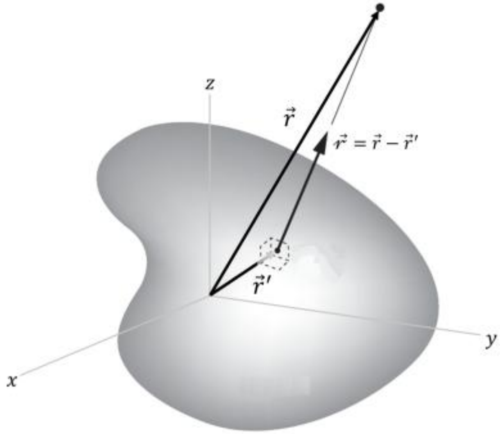
$$\iiint_{\mathcal{V}} \nabla\cdot\vec{E}\, d\tau = \frac{q}{4\pi\epsilon_0}\iiint_{\mathcal{V}} 4\pi\delta^3(r)\, d\tau = \frac{q}{\epsilon_0}$$

R.H.S.:

Since we can have arbitrary shape of the closed surface, the integral always depends on the net charges enclosed within the closed surface as

$$\oiint_{\mathcal{S}} \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0} \cdots\cdots \text{Gauss's law}$$

(4) Arbitrary charge distributions



$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \rho(r') d^3r'$$

$$\nabla \cdot \vec{E} = \frac{1}{4\pi\epsilon_0} \int \nabla \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \rho(r')\, d^3r'$$

$$= \frac{1}{4\pi\epsilon_0} \int 4\pi\delta^3(r - r')\rho(r')\, d^3r'$$

$$= \frac{\rho(r)}{\epsilon_0}$$

The volume integral becomes

$$\iiint_{\mathcal{V}} \nabla \cdot \vec{E}\, d\tau = \iiint_{\mathcal{V}} \frac{\rho(r)}{\epsilon_0} d\tau = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho(r)\, d\tau \Rightarrow \nabla \cdot \vec{E} = \frac{\rho(r)}{\epsilon_0}$$

(5) Thus, Gauss's law for arbitrary charge distributions in integral form:

$$\oint_{\mathcal{S}} \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho(r)\, d\tau$$

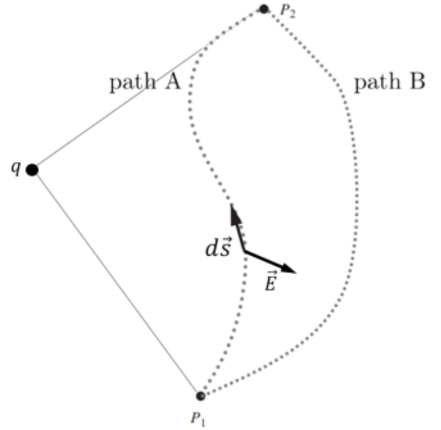
in differential form:

$$\nabla \cdot \vec{E} = \frac{\rho(r)}{\epsilon_0}$$

(6) For a given ρ, the electrostatic field $\vec{E}$ is not uniquely determined by

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

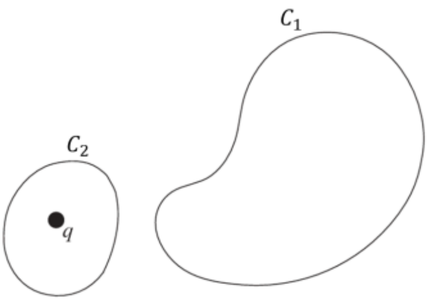
According to the Helmholtz theorem, we need another condition: we can look at the line integral of $\vec{E}$ around a closed path in this field.



The line integral of the field along path A from the point P_1 to the point P_2

$$\begin{aligned}\int_{P_1}^{P_2} \vec{E} \cdot d\vec{s} &= \int_{P_1}^{P_2} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \cdot \left(dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta\, d\phi\hat{\phi}\right) \\ &= \int_{P_1}^{P_2} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr \\ &= \frac{1}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\end{aligned}$$

The line integral for any given electrostatic field $\vec{E}$ has the same value for all paths from P_1 to P_2. Thus, the line integral around any closed path in an electrostatic field is zero.



Then, using Stokes' theorem, we get

$$\oint_{\mathcal{C}} \vec{E} \cdot d\vec{l} = \iint_{\mathcal{S}} \left(\nabla \times \vec{E}\right) \cdot d\vec{a} = 0 \Rightarrow \nabla \times \vec{E} = 0$$

(7) Thus, as $\vec{E}$ goes to zero at infinitely, we have

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \cdots\cdots \text{ Gauss's law}$$

$$\nabla \times \vec{E} = 0 \cdots\cdots \text{ no name}$$

and $\vec{E}$ is uniquely determined if ρ is given.

PROOF:

Suppose both equations are satisfied by two different fields $\vec{E}_1$ and $\vec{E}_2$.

$$\vec{D} = \vec{E}_1 - \vec{E}_2$$

$$\nabla \times \vec{D} = \nabla \times \vec{E}_1 - \nabla \times \vec{E}_2 = 0 \Rightarrow \vec{D} = \nabla f$$

$$\nabla \cdot \vec{D} = \nabla \cdot \vec{E}_1 - \nabla \cdot \vec{E}_2 = \frac{\rho}{\epsilon_0} - \frac{\rho}{\epsilon_0} = 0 \Rightarrow \nabla \cdot \nabla f = \nabla^2 f = 0$$

Since $\vec{E}_1$ and $\vec{E}_2$ go to zero at infinity (boundary), and

$$\vec{D} = \vec{E}_1 - \vec{E}_2 = \nabla f = 0$$

at the boundary. Thus, f takes on some constant value f_0 at the boundary. Since Laplace's equation allows no local maxima or minima—all extrema occur on the boundaries. So f must be the value f_0 everywhere. Hence

$$\vec{D} = \nabla f = 0 \text{ and } \vec{E}_1 = \vec{E}_2$$

■

EXAMPLES:

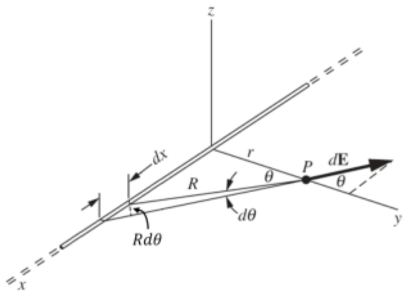
1. Line charge distribution

   A long, straight, charged wire that carries a uniform line charge λ. What is the electric field?

   ANSWER:

- Method I:

  The contribution of the charge dq to the y component of the electric field at P is



$$dE_y = \frac{dq}{4\pi\epsilon_0 R^2}\cos\theta = \frac{\lambda dx}{4\pi\epsilon_0 R^2}\cos\theta$$

Since

$$r = R\cos\theta$$

$$dx\cos\theta = Rd\theta \Rightarrow dx = \frac{Rd\theta}{\cos\theta} = \frac{rd\theta}{\cos^2\theta}$$

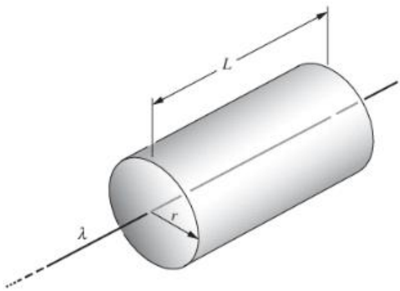
we obtain

$$dE_y = \frac{\lambda}{4\pi\epsilon_0}\left(\frac{\cos\theta}{r}\right)^2 \frac{rd\theta}{\cos^2\theta}\cos\theta = \frac{\lambda}{4\pi\epsilon_0 r}\cos\theta\, d\theta$$

Then, we have

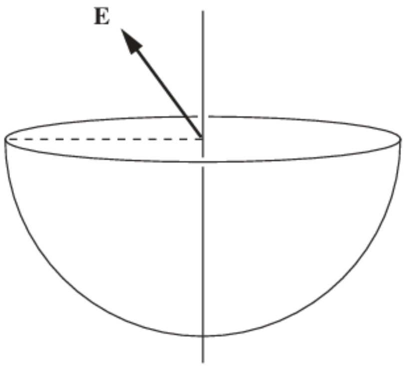
$$E_y = \int_{-\pi/2}^{\pi/2} \frac{\lambda}{4\pi\epsilon_0 r}\cos\theta\, d\theta = \frac{\lambda}{2\pi\epsilon_0 r}$$

- Method II:
  Using Gauss's law



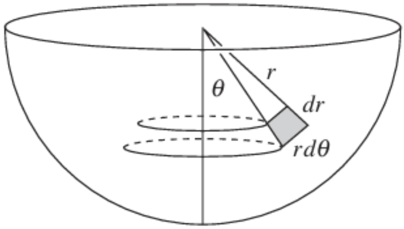
$$\oint_S \vec{E}\cdot d\vec{a} = \frac{q}{\epsilon_0} \Rightarrow E\cdot 2\pi rL = \frac{1}{\epsilon_0}L\lambda \Rightarrow \vec{E} = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{r}\hat{r}$$

2. Hemisphere charge distribution
   A solid hemisphere has radius R and uniform charge density ρ. Find the electric field at the center.



ANSWER:

Consider a tiny piece of the ring, with charge dq.



Because the horizontal component cancels with the horizontal component, only the vertical component survives.

$$dE_y = \frac{\rho}{4\pi\epsilon_0 r^2}\left(2\pi r^2 \sin\theta \, dr d\theta\right)\cos\theta = \frac{\rho}{2\epsilon_0}\sin\theta\cos\theta \, dr d\theta$$

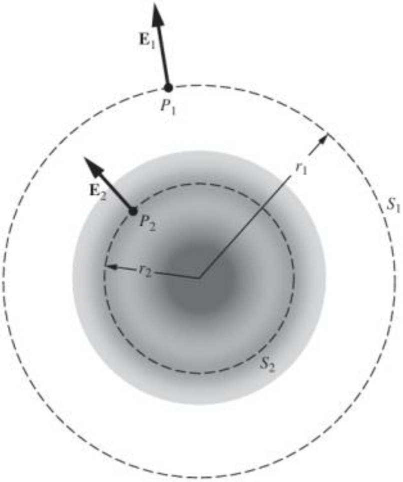
$$E_y = \frac{\rho}{2\epsilon_0}\int_0^R \int_0^{\pi/2} \sin\theta\cos\theta \, dr d\theta$$

$$= \frac{\rho}{2\epsilon_0}\int_0^R dr \int_0^{\pi/2} \sin\theta \, d\left(\sin\theta\right)$$

$$= \frac{\rho}{2\epsilon_0}\cdot r\Big|_0^R \cdot \frac{\sin^2\theta}{2}\Bigg|_0^{\pi/2}$$

$$= \frac{\rho R}{4\epsilon_0}$$

3. Spherical charge distribution (solid sphere)
   A spherical charge distribution has a density ρ that is constant from $r = 0$ out to $r = R$ and is zero beyond. What is the electric field?



ANSWER:

Using Gauss's law

For $r \geq R$,

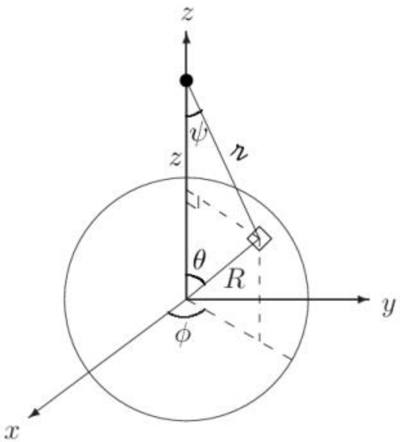
$$\oint_{\mathcal{S}} \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0} \Rightarrow E \cdot 4\pi r^2 = \frac{1}{\epsilon_0}\frac{4}{3}\pi R^3 \rho \Rightarrow \vec{E} = \frac{\rho}{3\epsilon_0}\frac{R^3}{r^2}\hat{r}$$

For $r \leq R$,

$$\oint_{\mathcal{S}} \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0} \Rightarrow E \cdot 4\pi r^2 = \frac{1}{\epsilon_0}\frac{4}{3}\pi r^3 \rho \Rightarrow \vec{E} = \frac{\rho}{3\epsilon_0} r\hat{r}$$

4. Spherical charge distribution (shell)

A spherical charge distribution has a density σ that is constant at $r = R$ and is zero elsewhere. What is the electric field?



ANSWER:

- Method I:

$$\mathscr{r}^2 = R^2 + z^2 - 2Rz\cos\theta$$

$$E_z = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma\cos\psi}{\mathscr{r}^2} R^2 \sin\theta \, d\theta d\phi$$

$$= \frac{\sigma R^2}{4\pi\epsilon_0}\int_0^{\pi} \frac{1}{\mathscr{r}^2}\frac{z - R\cos\theta}{\mathscr{r}} \sin\theta\, d\theta \underbrace{\int_0^{2\pi} d\phi}_{=2\pi}$$

$$= \frac{\sigma R^2}{2\epsilon_0}\int_0^{\pi} \frac{z - R\cos\theta}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}} \sin\theta\, d\theta$$

Let $u = \cos\theta$. We obtain

$$E_z = -\frac{\sigma R^2}{2\epsilon_0}\int_1^{-1} \frac{z - Ru}{(R^2 + z^2 - 2Rzu)^{3/2}} du$$

$$= \frac{\sigma R^2}{2\epsilon_0}\int_{-1}^{1} \frac{z - Ru}{(R^2 + z^2 - 2Rzu)^{3/2}} du$$

$$= \frac{\sigma R^2}{2\epsilon_0}\left[\frac{1}{z^2}\frac{zu - R}{(R^2 + z^2 - 2Rzu)^{1/2}}\right]_{-1}^{1}$$

$$= \frac{\sigma R^2}{2\epsilon_0 z^2}\left[\frac{z - R}{|z - R|} + 1\right]$$

Finally, we get

$$E_z = \begin{cases} \dfrac{\sigma R^2}{\epsilon_0 z^2}, & z \geq R \\ 0 \quad, & z < R \end{cases}$$

- Method II:

Using Gauss's law

For $r \geq R$,

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{q}{\epsilon_0} \Rightarrow E\cdot 4\pi r^2 = \frac{1}{\epsilon_0} 4\pi R^2\sigma \Rightarrow \vec{E} = \frac{\sigma}{\epsilon_0}\frac{R^2}{r^2}\hat{r}$$

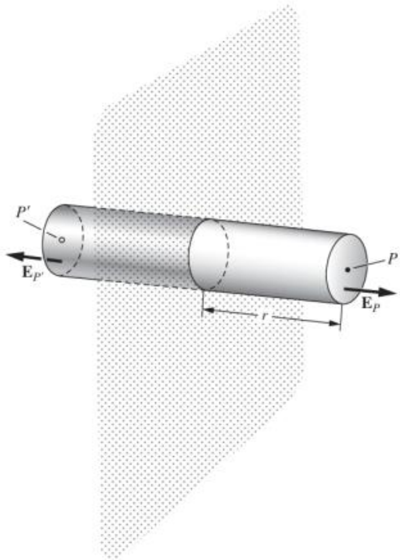
For $r < R$,

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{q}{\epsilon_0} \Rightarrow E\cdot 4\pi r^2 = 0 \Rightarrow \vec{E} = 0$$

5. Infinite flat sheet charge distribution

Consider a flat sheet, infinite in extent, with the constant surface charge density σ. Find the electric field.

ANSWER:



Using Gauss's law

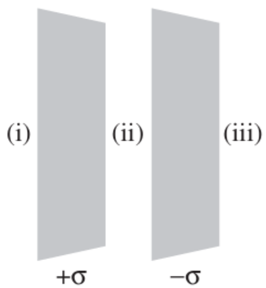
$$E_P \cdot A + E_{P'} \cdot A = \frac{\sigma A}{\epsilon_0}$$

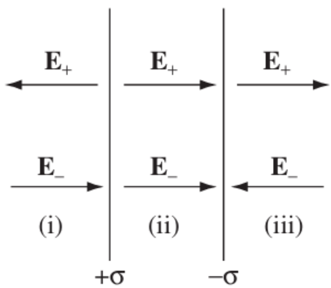
Since $E_P = E_{P'}$, we obtain

$$E_P = \frac{\sigma}{2\epsilon_0}$$

6. Two infinite parallel planes carry equal but opposite uniform charge densities $\pm\sigma$. Find the electric field.

ANSWER:





$$E_+ = \frac{\sigma}{2\epsilon_0}, \qquad E_- = -\frac{\sigma}{2\epsilon_0}$$

Therefore,

in region (i) $E = 0$

in region (ii) $E = E_+ + E_- = \frac{\sigma}{\epsilon_0}$

in region (iii) $E = 0$
The field between the plates is σ/ϵ_0, and points to the right; elsewhere it is zero.

2-2 Electric Potential

A. POTENTIAL

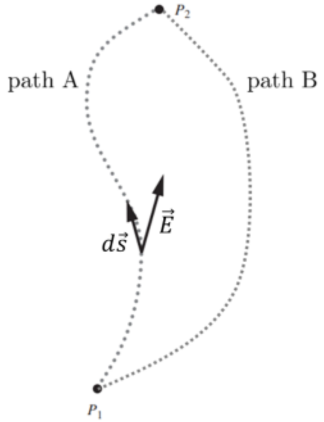
(1) Using Stokes' theorem, we obtain

$$\oint_{C} \vec{E} \cdot d\vec{s} = \iint_{S} \left(\nabla \times \vec{E}\right) \cdot d\vec{a}$$

Since the electrostatic field is an irrotational field, i.e., $\nabla \times \vec{E} = 0$, thus, we have

$$\oint_{C} \vec{E} \cdot d\vec{s} = 0$$

Consider two paths A and B:



$$\underbrace{\int_{P_1}^{P_2} \vec{E} \cdot d\vec{s}}_{\text{path A}} + \underbrace{\int_{P_2}^{P_1} \vec{E} \cdot d\vec{s}}_{\text{path B}} = 0$$

$$\underbrace{\int_{P_1}^{P_2} \vec{E} \cdot d\vec{s}}_{\text{path A}} - \underbrace{\int_{P_1}^{P_2} \vec{E} \cdot d\vec{s}}_{\text{path B}} = 0$$

Thus, the line integral is independent of path, i.e., the work done by the electric field is independent of path,

$$\int_{P_1}^{P_2} \vec{E} \cdot d\vec{s} = \boxed{-}\Big(\varphi(P_2) - \varphi(P_1)\Big)$$

where φ is a scalar function and the minus sign indicates that the increase of the work done by the electric field is equal to the decrease of the function.

Then, we can define a scalar quantity $\Delta\varphi$ without specifying any particular path:

$$\Delta\varphi \equiv -\int_{P_1}^{P_2} \vec{E}\cdot d\vec{s} \cdots\cdots \text{electric potential difference}$$

Since the line integral of a gradient is given by [c.f.1-2]

$$\int_{P_1}^{P_2} \nabla\varphi \cdot d\vec{s} = \varphi(P_2) - \varphi(P_1)$$

thus, we obtain

$$\vec{E} = -\nabla\varphi$$

The electric field points from a region of greater potential toward a region of lesser potential, whereas the vector $\nabla\varphi$ is defined so that it points in the direction of increasing φ.

(2) Suppose we hold P_1 fixed at some reference position. Then $\Delta\varphi$ becomes a function of P_2 only, i.e.,

$$\varphi(r) = -\int_{P_1}^{r} \vec{E}\cdot d\vec{s} \cdots\cdots \text{(a)}$$

Once the vector field $\vec{E}$ is given, the potential function $\varphi(r)$ is determined, except for an arbitrary additive constant allowed by the arbitrariness in the choice of P_1. Thus, $\varphi(r)$ is a scalar function of position, i.e., its value at a point is simply a number and has no direction associated with it.

EXAMPLES:

1. A long, straight, charged wire that carries a uniform line charge λ. What is the potential?

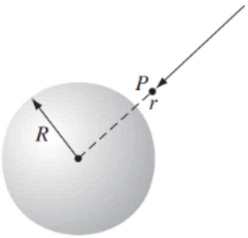
   ANSWER:

   From Gauss's law, the field outside is

   $$\vec{E} = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{r}\hat{r}$$

$$\begin{aligned}
\varphi(r) &= -\int_{r_0}^{r} \vec{E} \cdot d\vec{s} \\
&= -\frac{\lambda}{2\pi\epsilon_0}\int_{r_0}^{r} \frac{1}{r'}\hat{r} \cdot \left(dr'\hat{r} + r'd\theta\hat{\theta} + r'\sin\theta\, d\phi\hat{\phi}\right) \\
&= -\frac{\lambda}{2\pi\epsilon_0}\int_{r_0}^{r} \frac{1}{r'}dr' \\
&= -\frac{\lambda}{2\pi\epsilon_0}\ln r'\Big|_{r_0}^{r} \\
&= -\frac{\lambda}{2\pi\epsilon_0}\ln r + \text{constant}
\end{aligned}$$

2. Find the potential inside and outside a spherical shell of radius R that carries a uniform surface charge σ. Set the reference point at infinity.



ANSWER:

From Gauss's law, the field outside is

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} = \frac{1}{4\pi\epsilon_0}\frac{4\pi R^2\sigma}{r^2}\hat{r} = \frac{\sigma}{\epsilon_0}\frac{R^2}{r^2}\hat{r}$$

For points outside the sphere ($r \geq R$):

$$\begin{aligned}
\varphi(r) &= -\int_{\infty}^{r} \vec{E} \cdot d\vec{s} \\
&= -\int_{\infty}^{r} \frac{\sigma}{\epsilon_0}\frac{R^2}{r'^2}\hat{r}' \cdot \left(dr'\hat{r} + r'd\theta\hat{\theta} + r'\sin\theta\, d\phi\hat{\phi}\right) \\
&= -\frac{\sigma R^2}{\epsilon_0}\int_{\infty}^{r} \frac{1}{r'^2}dr' \\
&= \frac{\sigma R^2}{\epsilon_0}\frac{1}{r'}\Big|_{\infty}^{r} \\
&= \frac{\sigma R^2}{\epsilon_0 r}
\end{aligned}$$

The field inside is

$\vec{E} = 0$

For points inside the sphere $(r < R)$, we must break the integral into two pieces:

$$\varphi(r) = -\int_{\infty}^{R} \frac{\sigma}{\epsilon_0} \frac{R^2}{r'^2} dr' - \int_{R}^{r} 0\, dr' = \frac{\sigma R^2}{\epsilon_0} \frac{1}{r'}\bigg|_{\infty}^{R} = \frac{\sigma R}{\epsilon_0}$$

The potential is not zero inside the shell, even though the field is.

3. Find the potential inside and outside a uniformly charged solid sphere whose radius is R and total charge is q.

ANSWER:

From Gauss's law, the field outside is

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

For points outside the sphere $(r \geq R)$:

$$\varphi(r) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{s}$$

$$= -\int_{\infty}^{r} \frac{1}{4\pi\epsilon_0} \frac{q}{r'^2} \hat{r}' \cdot \left(dr'\hat{r} + r' d\theta\hat{\theta} + r' \sin\theta\, d\phi\hat{\phi}\right)$$

$$= -\frac{q}{4\pi\epsilon_0} \int_{\infty}^{r} \frac{1}{r'^2} dr'$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{r'}\bigg|_{\infty}^{r}$$

$$= \frac{q}{4\pi\epsilon_0 r}$$

From Gauss's law, the field inside is

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{R^3} r\hat{r}$$

For points inside the sphere $(r < R)$, we must break the integral into two pieces:

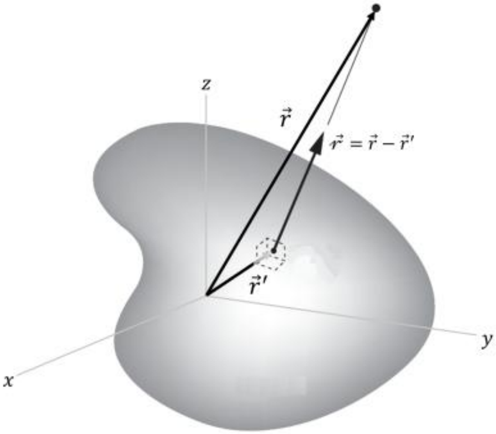
$$\varphi(r) = -\int_{\infty}^{R} \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}\,dr' - \int_{R}^{r} \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r'\,dr'$$
$$= \frac{q}{4\pi\epsilon_0 R} - \frac{q}{4\pi\epsilon_0 R^3}\frac{1}{2}r'^2\Big|_R^r$$
$$= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R} - \frac{r^2}{2R^3} + \frac{1}{2R}\right)$$
$$= \frac{q}{4\pi\epsilon_0}\frac{1}{2R}\left(3 - \frac{r^2}{R^2}\right)$$

B. POTENTIAL OF A CHARGE DISTRIBUTION

(1) From equation (a), we have the potential as

$$\varphi(r) = -\int_{P_1}^{r} \vec{E}\cdot d\vec{s}$$

For a continuous charge distribution, the field is



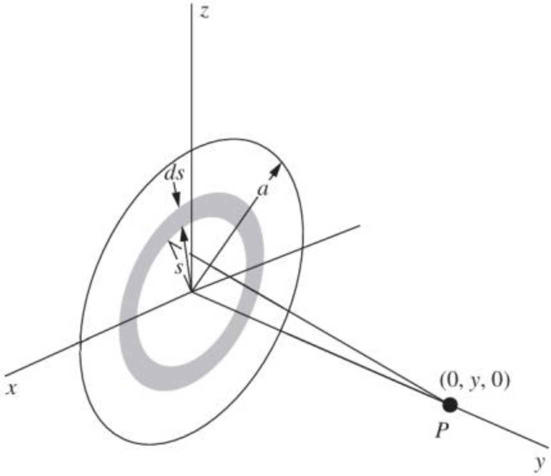
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int_V \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\rho(r')\,d\tau'$$

Thus, we obtain

$$\varphi(r) = -\frac{1}{4\pi\epsilon_0}\int_{\infty}^{\mathscr{r}} \frac{\rho(r')}{\mathscr{r}'^2}\hat{\mathscr{r}}\cdot\left(d\mathscr{r}'\hat{\mathscr{r}} + \mathscr{r}'d\theta\hat{\theta} + \mathscr{r}'\sin\theta\,d\phi\hat{\phi}\right)d\tau'$$
$$= -\frac{1}{4\pi\epsilon_0}\int_{\infty}^{\mathscr{r}} \frac{\rho(r')}{\mathscr{r}'^2}d\mathscr{r}'d\tau'$$
$$= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r')}{\mathscr{r}}d\tau'$$

EXAMPLES:

1. Find the potential on the axis of a uniformly charged disk of radius a.



ANSWER:

$r^2 = y^2 + s^2$

$$\varphi(r) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(r')}{r} da'$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{\sqrt{y^2+s^2}} s ds d\theta$$

$$= \frac{\sigma}{4\pi\epsilon_0}\int_0^a \frac{sds}{\sqrt{y^2+s^2}} \underbrace{\int_0^{2\pi} d\theta}_{=2\pi}$$

$$= \frac{\sigma}{2\epsilon_0}\sqrt{y^2+s^2}\Big|_0^a$$

$$= \frac{\sigma}{2\epsilon_0}\left[\sqrt{y^2+a^2} - y\right]$$

For $y \gg a$, we can approximate

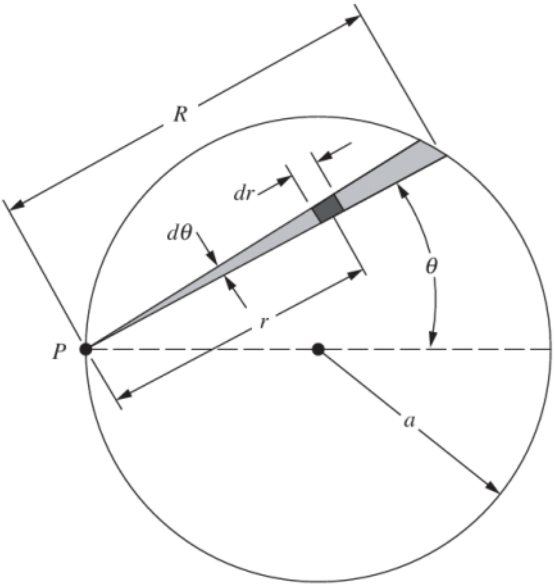
$$\sqrt{y^2+a^2} - y = y\sqrt{1+\frac{a^2}{y^2}} - y = y\left[1 + \frac{1}{2}\frac{a^2}{y^2} + \cdots\right] - y \approx \frac{a^2}{2y}$$

Thus, we obtain

$$\varphi(r) = \frac{\sigma}{2\epsilon_0}\frac{a^2}{2y} = \frac{\sigma a^2}{4\epsilon_0 y} = \frac{q}{4\pi\epsilon_0 y}$$

2. Find the potential on the rim of a uniformly charged disk of

radius a.



ANSWER:

$$R = 2a\cos\theta$$

$$\varphi(r) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(r')}{\mathscr{r}}da'$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{r} r dr d\theta$$

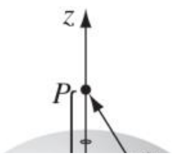
$$= \frac{\sigma}{4\pi\epsilon_0}\int_0^R dr\, d\theta$$

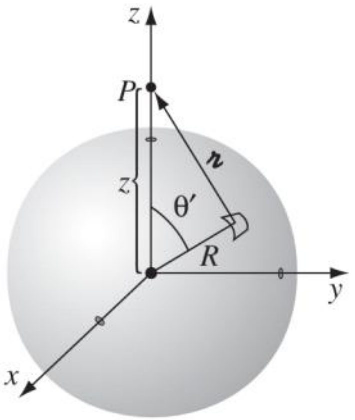
$$= \frac{\sigma}{4\pi\epsilon_0}\int R\, d\theta$$

$$= \frac{\sigma}{4\pi\epsilon_0}\int_{-\pi/2}^{\pi/2} 2a\cos\theta\, d\theta$$

$$= \frac{\sigma a}{\pi\epsilon_0}$$

3. Find the potential of a uniformly charged spherical shell of radius R.





ANSWER:

$$\mathscr{r}^2 = R^2 + z^2 - 2Rz\cos\theta'$$

$$\varphi(r) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(r')}{\mathscr{r}} da'$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}} R^2 \sin\theta'\, d\theta' d\phi'$$

Since

$$\int_0^{2\pi} d\phi' = 2\pi$$

$$\int_0^{\pi} \frac{\sin\theta'\, d\theta'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}} = \frac{1}{Rz}\sqrt{R^2 + z^2 - 2Rz\cos\theta'}\Big|_0^{\pi}$$

$$= \frac{1}{Rz}(R + z - |R - z|)$$

$$= \begin{cases} 2/z\,, & \text{if } z \geq R \\ 2/R\,, & \text{if } z < R \end{cases}$$

Thus, we obtain

$$\varphi(r) = \begin{cases} \dfrac{\sigma R^2}{4\pi\epsilon_0} \cdot 2\pi \cdot \dfrac{2}{z} \\ \dfrac{\sigma R^2}{4\pi\epsilon_0} \cdot 2\pi \cdot \dfrac{2}{R} \end{cases} = \begin{cases} \dfrac{\sigma R^2}{\epsilon_0 z}, & z \geq R \\ \dfrac{\sigma R}{\epsilon_0}\ , & z < R \end{cases}$$

4. Find the potential inside a uniformly charged solid sphere whose radius is R and total charge is q.

ANSWER:

$$\mathscr{r}^2 = r^2 + z^2 - 2rz\cos\theta'$$

$$\varphi(r) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r')}{\mathscr{r}}\,d\tau'$$
$$= \frac{1}{4\pi\epsilon_0}\int \frac{\rho}{\sqrt{r'^2 + z^2 - 2r'z\cos\theta'}} r^2 \sin\theta'\,dr'd\theta'd\phi'$$

Since

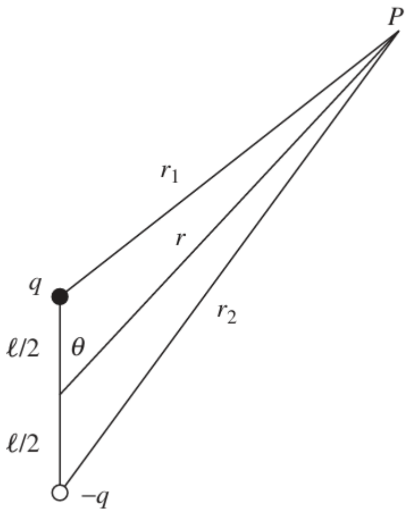
$$\int_0^{2\pi} d\phi' = 2\pi$$

$$\int_0^{\pi} \frac{\sin\theta'\,d\theta'}{\sqrt{r'^2 + z^2 - 2r'z\cos\theta'}} = \frac{1}{r'z}\sqrt{r'^2 + z^2 - 2r'z\cos\theta'}\Big|_0^{\pi}$$
$$= \frac{1}{r'z}(r' + z - |r' - z|)$$
$$= \begin{cases} 2/z\,, & \text{if } r < z \\ 2/r\,, & \text{if } r > z \end{cases}$$

Thus, we obtain

$$\varphi(r) = \frac{\rho}{4\pi\epsilon_0} 4\pi\left[\int_0^z \frac{1}{z} r'^2\,dr' + \int_z^R \frac{1}{r} r^2\,dr\right]$$
$$= \frac{\rho}{\epsilon_0}\left[\frac{1}{z}\frac{z^3}{3} + \frac{1}{2}R^2 - \frac{1}{2}z^2\right]$$
$$= \frac{q}{4\pi\epsilon_0}\frac{3}{R^3}\frac{1}{2}\left(R^2 - \frac{z^2}{3}\right)$$
$$= \frac{q}{4\pi\epsilon_0}\frac{1}{2R}\left(3 - \frac{z^2}{R^2}\right)$$

5. Find the potential of a dipole.



ANSWER:

$$\varphi = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{r^2 + \frac{l^2}{4} - rl\cos\theta}} - \frac{1}{\sqrt{r^2 + \frac{l^2}{4} - rl\cos(\pi-\theta)}}\right)$$

In the limit $r \gg l$, we have

$$\frac{1}{\sqrt{r^2 + \frac{l^2}{4} - rl\cos\theta}} = \frac{1}{r}\frac{1}{\sqrt{1 + \frac{l^2}{4r^2} - \frac{l}{r}\cos\theta}}$$

$$= \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{l^2}{4r^2} - \frac{l}{r}\cos\theta\right) + \frac{3}{8}\left(\frac{l^2}{4r^2} - \frac{l}{r}\cos\theta\right)^2\right]$$

$$= \frac{1}{r}\left[1 + \frac{l}{2r}\cos\theta + \left(\frac{l^2}{r^2}\right)\left(\frac{3}{8}\cos^2\theta - \frac{1}{8}\right) + \cdots\right]$$

Thus, we obtain

$$\varphi = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{r^2 + \frac{l^2}{4} - rl\cos\theta}} - \frac{1}{\sqrt{r^2 + \frac{l^2}{4} - rl\cos(\pi-\theta)}}\right)$$

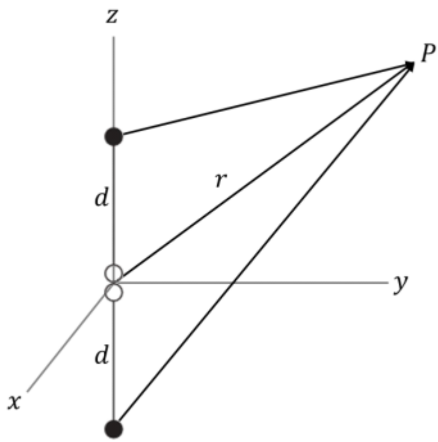
$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r}\left(1 + \frac{l}{2r}\cos\theta\right) - \frac{1}{r}\left(1 + \frac{l}{2r}\cos(\pi-\theta)\right)\right]$$

$$= \frac{ql}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}$$

The quantity ql is called the electric dipole moment denoted by $p = ql$.
The field is

$$\begin{aligned}\vec{E} &= -\nabla\varphi \\ &= -\frac{\partial\varphi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}\hat{\phi} \\ &= \frac{ql}{4\pi\epsilon_0}\frac{2\cos\theta}{r^3}\hat{r} + \frac{1}{r}\frac{ql}{4\pi\epsilon_0}\frac{\sin\theta}{r^2}\hat{\theta} \\ &= \frac{ql}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)\end{aligned}$$

6. Find the potential of a linear quadrupole.



ANSWER:

$$\varphi = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r} - \frac{1}{r} + \frac{1}{r_2}\right) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{2}{r} + \frac{1}{r_2}\right)$$

In the limit $r \gg l$, we obtain

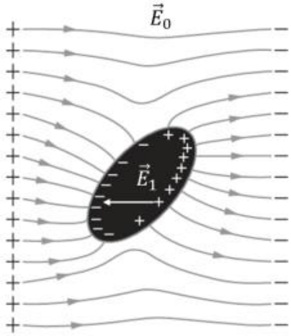
$$\begin{aligned}\varphi &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{r^2 + d^2 - 2rd\cos\theta}} - \frac{1}{\sqrt{r^2 + d^2 - 2rd\cos(\pi - \theta)}}\right) \\ &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r}\left(1 + \frac{d}{r}\cos\theta + \left(\frac{d^2}{r^2}\right)\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\right) - \frac{2}{r}\right. \\ &\qquad \left. + \frac{1}{r}\left(1 + \frac{d}{r}\cos(\pi - \theta) + \left(\frac{d^2}{r^2}\right)\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\right)\right] \\ &= \frac{2qd^2}{4\pi\epsilon_0}\frac{3\cos^2\theta - 1}{2r^3}\end{aligned}$$

The quantity $2qd^2$ is called the electric quadrupole moment denoted by $Q = 2qd^2$.

2-3 Conductors

A. ELECTROSTATIC INDUCTION

(1) Consider a conductor in an external electric field $\vec{E}_0$



The field will drive free positive charges to the right, and negative ones to the left. These induced charges would be redistributed due to the Coulonb's force between other charges.

(2) Inside a conductor, the field of the induced charges, $\vec{E}_1$ has the opposite direction of the external field $\vec{E}_0$, cancel the original field. Charge will continue to flow until this cancellation is complete, and the resultant field inside the conductor is precisely zero.

Let $\mathcal{S}$ be a closed surface inside the conductor. By Gauss's law

$$\oint_{\mathcal{S}} \vec{E} \cdot d\vec{a} = \frac{\rho}{\epsilon_0} = 0 \Rightarrow \rho = 0$$

This implies that no net charge inside the conductor, and the charge must all reside on the surface.

Since

$$\vec{E} = -\nabla \varphi = 0 \Rightarrow \varphi = \text{constant}$$

a conductor is an equipotential at all points inside.

(3) Outside a conductor, if there were a tangential component to the field at the surface, the charge would again move to eliminate it.

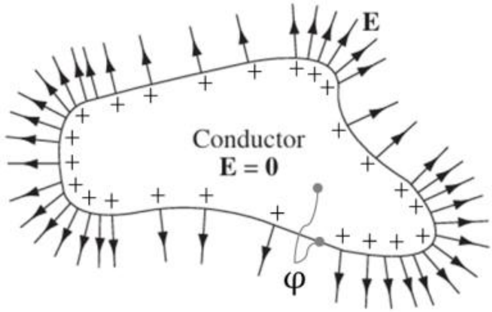
This implies that, in equilibrium, at the surface, the field $\vec{E}$ is normal to the surface

$$\vec{E} = E_{\perp}\hat{n}$$

Since

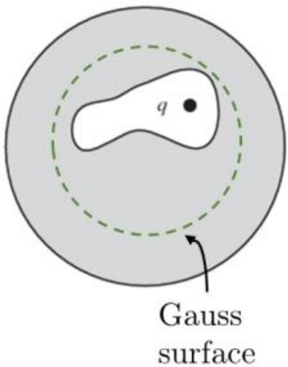
$$E_{\parallel} = -\nabla\varphi_{\text{surface}} = 0 \Rightarrow \varphi = \text{constant}$$

a conductor is an equipotential on the surface.



B. SCREENING AND SHIELDING

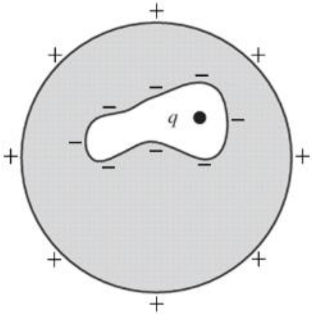
(1) Consider an uncharged spherical conductor containing a cavity with any shape and put a charge in somewhere within the cavity.



Since inside the conductor the net field must be zero, thus we have

$$\oint_S \underbrace{\vec{E}}_{=0} \cdot d\vec{a} = \frac{q_{\text{enc}}}{\epsilon_0} = 0 \Rightarrow q_{\text{enc}} = q + q_{\text{ind}} = 0 \Rightarrow q_{\text{ind}} = -q$$

The charge $-q$ is induced on the inner surface. Thus, there must be a charge $+q$ on the outer surface to maintain the zero net field inside the conductor.

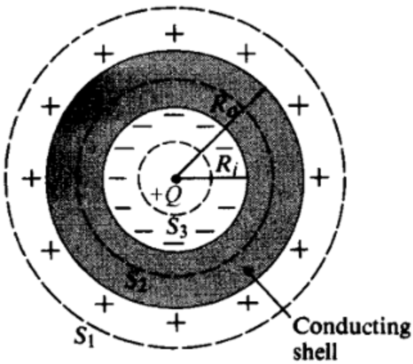


Since the asymmetrical influence of the point charge $+q$ is negated by that of the induced charge $-q$ on the inner surface, the charge $+q$ is therefore distributed uniformly over the surface.
Thus the shape of the cavity is regardless and the field outside the conductor is

$$\oint_S \vec{E} \cdot d\vec{a} = E4\pi r^2 = \frac{q}{\epsilon_0} \Rightarrow \vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$$

EXAMPLES:

1. A positive point charge Q is at the center of a spherical conducting shell of an inner radius R_i and an outer radius R_o. Determine $\vec{E}$ and φ as functions of the radial distance R.



ANSWER:

- For $R > R_o$

$$\oint_{S_1} \vec{E} \cdot d\vec{a} = E4\pi R^2 = \frac{Q}{\epsilon_0} \Rightarrow E = \frac{Q}{4\pi\epsilon_0 R^2}$$

$$\varphi(R) = -\int_\infty^R E\, dR = -\int_\infty^R \frac{Q}{4\pi\epsilon_0 R^2} dR = \frac{Q}{4\pi\epsilon_0 R}$$

- For $R_i < R < R_o$

$E = 0$

Since the conducting shell is equipotential, hence $\varphi(R) = \varphi(R_o)$

$$\varphi(R_o) = \frac{Q}{4\pi\epsilon_0 R_o}$$

- For $R < R_i$

$$\oint_{S_1} \vec{E} \cdot d\vec{a} = E4\pi R^2 = \frac{Q}{\epsilon_0} \Rightarrow E = \frac{Q}{4\pi\epsilon_0 R^2}$$

$$\varphi(R) = -\int E\, dR + C = \frac{Q}{4\pi\epsilon_0 R} + C$$

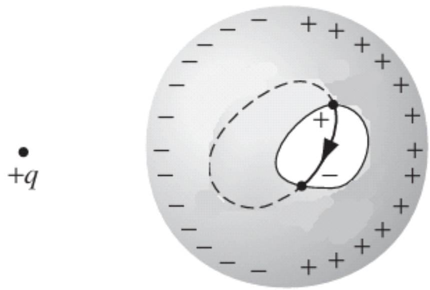
Since at $R = R_i$, we have $\varphi(R_i) = \varphi(R_o)$, that is,

$$\frac{Q}{4\pi\epsilon_0 R_i} + C = \frac{Q}{4\pi\epsilon_0 R_o} \Rightarrow C = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_o} - \frac{1}{R_i}\right)$$

Thus, we obtain

$$\varphi(R) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R} + \frac{1}{R_o} - \frac{1}{R_i}\right)$$

(2) Consider a conductor together with a point charge external to its volume.



The line integral is zero for any electrostatic field:

$$\oint_C \vec{E} \cdot d\vec{s} = \int_A^B \vec{E}_{\text{cavity}} \cdot d\vec{s} + \int_B^A \vec{E}_{\text{conductor}} \cdot d\vec{s} = 0$$

Since

$$\vec{E}_{\text{conductor}} = 0$$

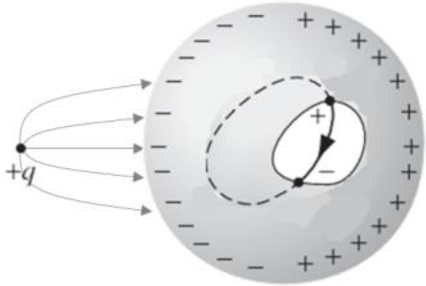
thus, we obtain

$$\int_A^B \vec{E}_{\text{cavity}} \cdot d\vec{s} = 0$$

We can conclude that

$$\vec{E}_{\text{cavity}} = 0$$

which implies that and any object placed inside the cavity is completely **screened** or **shielded** from the electrostatic effects of the exterior point charge q.



Thus, every electric field line that leaves q terminates on the outer surface of the conductor or goes off to infinity.

2-4 Capacitance

A. CAPACITANCE

(1) Conductors are used to store electric charge because their surfaces are easily accessible. We use the concept of capacitance to measure the quantitative capacity of any particular conductor to store charge, whether in isolation or in the presence of other conductors.

(2) Suppose we have two conductors,

Since φ is constant over a conductor, the potential difference between them is:

$$\Delta\varphi = \varphi(+) - \varphi(-) \geq 0$$

and

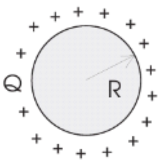
$$\Delta\varphi = -\int_{(-)}^{(+)} \vec{E} \cdot d\vec{s} = \frac{q}{4\pi\epsilon_0 \mathcal{V}}\left(\int^{(+)} \frac{r'}{r} d\tau' - \int^{(-)} \frac{r'}{r} d\tau'\right) = qZ$$

Hence the capacitance is defined by

$$C = \frac{q}{\Delta\varphi}$$

EXAMPLES:

1. Consider a uniformly charged conducting sphere with total charge Q and radius R. Find the capacity.



ANSWER:

The voltage difference at the radius R relative to infinity, where we define $\varphi(\infty) \equiv 0$ as ground, is given by

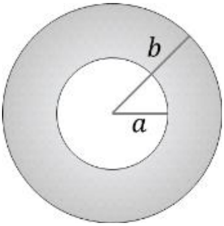
$$\Delta\varphi = \varphi(R) - \varphi(\infty) = \varphi(R) = \frac{Q}{4\pi\epsilon_0 R}$$

The capacitance C of the sphere is then defined by

$$C = \frac{Q}{\Delta\varphi} = 4\pi\epsilon_0 R$$

The larger the sphere, the greater its capacity to hold charge at a given $\Delta\varphi$.

2. Find the capacitance of two concentric spherical metal shells, with radii a and b.



ANSWER:

$$\Delta\varphi = -\int_b^a \vec{E}\cdot d\vec{s} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

The capacitance C is then defined by

$$C = \frac{Q}{\Delta\varphi} = \frac{1}{\frac{1}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)} = 4\pi\epsilon_0 \frac{ab}{a-b}$$

3. Find the capacitance per unit length of two coaxial metal cylindrical tubes, of radii a and b.

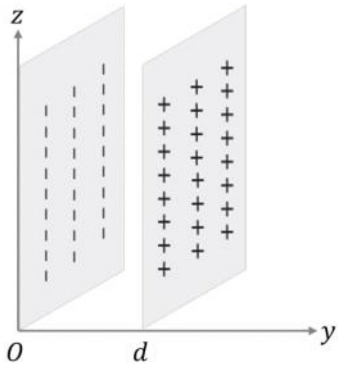
ASNWER:

$$\Delta\varphi = -\int_b^a \vec{E}\cdot d\vec{s} = -\frac{\lambda}{2\pi\epsilon_0}(\ln a - \ln b) = -\frac{Q}{2\pi\epsilon_0 L}\ln\frac{a}{b}$$

The capacitance C per unit length is then defined by

$$C = \frac{Q}{\Delta\varphi} = \frac{1}{\frac{1}{2\pi\epsilon_0 L}\ln\frac{b}{a}} = \frac{2\pi\epsilon_0}{\ln\frac{b}{a}}$$

4. Consider two parallel conducting plates with opposite charges $\pm Q$, separated by a distance d, which is called parallel-plate capacitor.



The total charge is

$Q = \sigma A$

The electric field between the plates is given by

$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{y}$$

The potential difference between the plates is then obtained as

$$\Delta\varphi = \varphi(+) - \varphi(-) = \frac{\sigma}{\epsilon_0}(y_2 - y_1) = \frac{\sigma}{\epsilon_0}d = \frac{\sigma A}{\epsilon_0}\frac{d}{A} = \frac{Qd}{\epsilon_0 A}$$

Hence the capacitance is

$$C = \frac{Q}{\Delta\varphi} = \frac{\epsilon_0 A}{d}$$

The larger the area A, and the smaller the separation d, the greater the capacity to hold charge at a given $\Delta\varphi$.

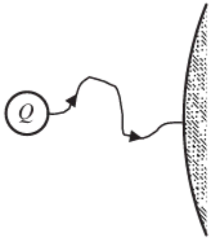
B. GROUND A CONDUCTOR

(1) The Earth has been considered as a very large charge reservoir that it has

$$C_{\text{Earth}} = 4\pi\epsilon_0 R \to \infty \text{ and } \varphi_{\text{Earth}} = \frac{Q}{C} \to 0$$

Thus, finite amounts of charge may be added to or taken from its surface without appreciably changing its potential from zero.

(2) We ground a conductor (fix its potential at zero) by connecting it to Earth using a fine conducting wire.

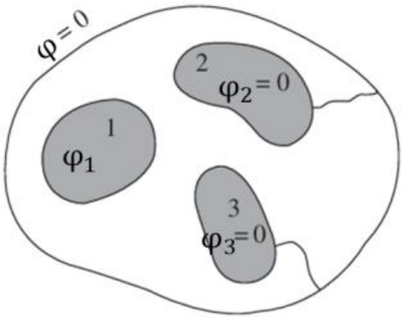


If the conductor initially has a net charge Q, the connection to ground causes this charge to spontaneously flow to the Earth. This lowers the potential energy of every charge on the conductor and leads to a final equilibrium state of $\varphi = 0$ for the conductor.

An initially uncharged conductor will draw charge up from ground due to Coulomb attraction in the presence of a nearby charge q.

C. COEFFICIENTS OF CAPACITANCE

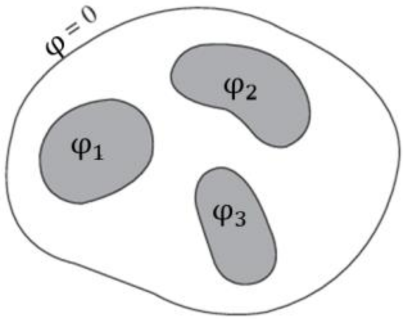
(1) Consider any number of conductors of some given configuration.



Connecting conductors 2 and 3 to the ground. We obtain

$$Q_1 = C_{11}\varphi_1$$
$$Q_2 = C_{21}\varphi_1$$
$$Q_3 = C_{31}\varphi_1$$

(2) If the potentials are φ_1, φ_2, and φ_3, none of them necessarily zero.



We obtain

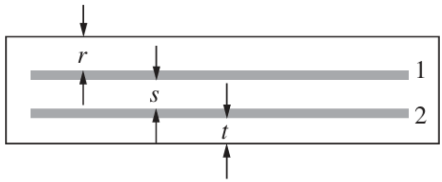
$$Q_1 = C_{11}\varphi_1 + C_{12}\varphi_2 + C_{13}\varphi_3$$
$$Q_2 = C_{21}\varphi_1 + C_{22}\varphi_2 + C_{23}\varphi_3$$
$$Q_3 = C_{31}\varphi_1 + C_{32}\varphi_2 + C_{33}\varphi_3$$

In matrix form

$$\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$$

EXAMPLES:

1. A flat metal box has zero potential. There are two plates in the box. Find the capacitance coefficients.

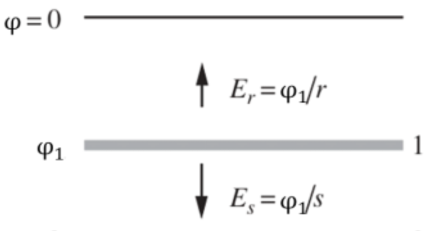


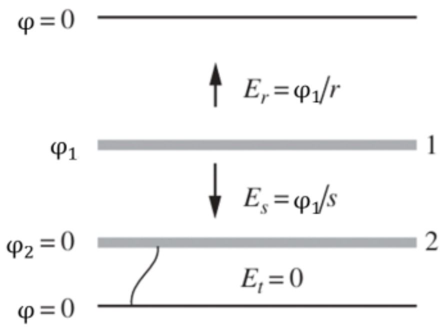
ANSWER:

With the potential of the box chosen to be zero, we can write, in general,

$$Q_1 = C_{11}\varphi_1 + C_{12}\varphi_2$$
$$Q_2 = C_{21}\varphi_1 + C_{22}\varphi_2$$

Consider the case where φ_2 is made equal to zero by connecting plate 2 to the box.





Then the fields in the three regions are $E_r = \varphi_1/r$, $E_s = \varphi_1/s$, and $E_t = 0$.

Gauss's law with a thin box completely surrounding plate 1 tells us that

$$E_r A + E_s A = \frac{Q_1}{\epsilon_0} \Rightarrow \left(\frac{\varphi_1}{r} + \frac{\varphi_1}{s}\right) A = \frac{Q_1}{\epsilon_0} \Rightarrow Q_1 = \epsilon_0 A \left(\frac{1}{r} + \frac{1}{s}\right) \varphi_1$$

Thus, we obtain

$$C_{11} = \epsilon_0 A \left(\frac{1}{r} + \frac{1}{s}\right)$$

Also, Gauss's law with a thin box completely surrounding plate 2 tells us that

$$-E_s A = \frac{Q_2}{\epsilon_0} \Rightarrow -\frac{\varphi_1}{s} A = \frac{Q_2}{\epsilon_0} \Rightarrow Q_2 = -\frac{\epsilon_0 A}{s} \varphi_1$$

Thus, we obtain

$$C_{21} = -\frac{\epsilon_0 A}{s}$$

Similarly, connecting plate 1 to the box, we can also obtain

$$C_{22} = \epsilon_0 A \left(\frac{1}{t} + \frac{1}{s}\right)$$

and

$$C_{12} = -\frac{\epsilon_0 A}{s}$$